L. 13

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Examples

① Canonical metric on quadrics. & their curvature

$$\tilde{M} = \underset{k}{\mathbb{R}} \times \underset{g}{\mathbb{R}^n} \qquad \tilde{g} = k \oplus g$$

$$\tilde{g} = k\,dx^{0\,2} + g_{\mu\nu}\,dx^\mu dx^\nu \ ; \qquad k = \pm 1$$

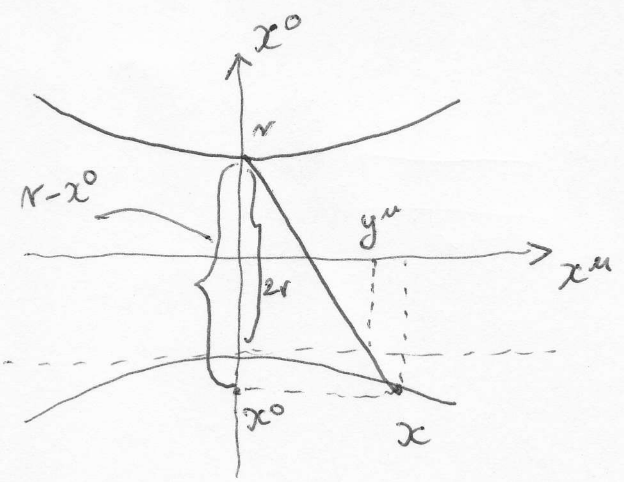
(signature of $g_{\mu\nu}$ can be arbitrary)

$$\Sigma_i = \{ (x^0, x^\mu) \in \tilde{M} : \ k(x^0)^2 + g_{\mu\nu} x^\mu x^\nu = k \cdot r^2 \}$$

$$\boxed{g_{\mu\nu} = \mathrm{diag}(1,\dots,1,-1,\dots,-1)}$$

$$K := \frac{1}{k r^2}$$

Stereographic projection: (you know it for S^n, so we do it for hyperboloids)



$$\frac{y^\mu}{2r} = \frac{x^\mu}{r - x^0}$$

$$\Downarrow$$

$$y^\mu = \frac{2x^\mu}{1 - \frac{x^0}{r}}$$

Claim

$$\boxed{\tilde{g}|_\Sigma = \frac{g_{\mu\nu}\,dy^\mu dy^\nu}{\left(1 + \frac{K}{4} g_{\mu\nu} y^\mu y^\nu\right)^2}}$$

Proof of the claim:

$$\begin{cases} g(x,x') = x x' = g_{\mu\nu} x^\mu x'^\nu \\ g(x,x) = |x|^2 \end{cases}$$

$$|y|^2 = \frac{4|x|^2}{\left(1 - \frac{x^0}{r}\right)^2}$$

$$k x^{02} + |x|^2 = k r^2 \Rightarrow \quad |x|^2 = k r^2 \left(1 - \frac{x^{02}}{r^2}\right)$$

$$\frac{K}{4}|y|^2 = \frac{1 - \frac{x^{02}}{r^2}}{\left(1 - \frac{x^0}{r}\right)^2} = \frac{1 + \frac{x^0}{r}}{1 - \frac{x^0}{r}}$$

$$\Rightarrow \quad \frac{x^0}{r} = \frac{\frac{K}{4}|y|^2 - 1}{\frac{K}{4}|y|^2 + 1} \quad \Rightarrow \quad 1 - \frac{x^0}{r} = \frac{2}{\frac{K}{4}|y|^2 + 1}$$

$$\Rightarrow \boxed{\frac{dx^0}{r} = \frac{K\, y\, dy}{\left(1 + \frac{K}{4}|y|^2\right)^2}}$$

$$\boxed{x^\mu = \frac{y^\mu}{1 + \frac{K}{4}|y|^2}}$$

$$dx^\mu = \frac{dy^\mu}{1 + \frac{K}{4}|y|^2} - \frac{y^\mu \frac{K}{2}\, y\, dy}{\left(1 + \frac{K}{4}|y|^2\right)^2}$$

$$\tilde{g}|_\Sigma = k\, dx^{02} + |dx|^2 =$$

$$= \frac{k}{r^2} \frac{(y\, dy)^2}{\left(1 + \frac{K}{4}|y|^2\right)^4} + \frac{|dy|^2}{\left(1 + \frac{K}{4}|y|^2\right)^2} - K \frac{(y\, dy)^2}{\left(1 + \frac{K}{4}|y|^2\right)^3}$$

$$+ \frac{K^2}{4} \frac{|y|^2 (y\, dy)^2}{\left(1 + \frac{K}{4}|y|^2\right)^4}$$

$$= \frac{|dy|^2}{\left(1 + \frac{K}{4}|y|^2\right)^2} + \frac{(y\, dy)^2}{\left(1 + \frac{K}{4}|y|^2\right)^4} \left[\cancel{K} - K\left(\cancel{1} + \cancel{\tfrac{K}{4}|y|^2}\right) + \cancel{\tfrac{K^2}{4}|y|^2} \right]$$

$$= \frac{|dy|^2}{\left(1 + \frac{K}{4}|y|^2\right)^2}$$

$\square$

Orthonormal frame:

$$\omega^\mu = \frac{dy^\mu}{1+\frac{K}{4} g_{\mu\nu} y^\mu y^\nu}$$

$$\tilde{g}_{|\Sigma} = g_{\mu\nu}\omega^\mu\omega^\nu$$

Structure equations:

(1) $d\omega^\mu + \Gamma^\mu{}_\varrho \wedge \omega^\varrho = 0$ no torsion

(2) $\cancel{dg_{\mu\nu}} - \Gamma_{\mu\nu} - \Gamma_{\nu\mu} = 0$ metricity

$$d\omega^\mu = -\frac{K}{2} y_\mu \, \omega^\nu \wedge \omega^\mu = \frac{K}{2}\left(y_\nu \omega^\mu - y^\mu \omega_\nu\right)\wedge\omega^\nu$$

$$\Downarrow$$

$$\boxed{\Gamma^\mu{}_\nu = -\frac{K}{2}\left(y_\nu\omega^\mu - y^\mu\omega_\nu\right)}$$

because it satisfies (1) ; $\Gamma_{\mu\nu} + \Gamma_{\nu\mu} = 0$ hence it also satisfies (2) + uniqueness of Levi-Civita!

Curvature

$$\Omega^\mu{}_\nu = d\Gamma^\mu{}_\nu + \Gamma^\mu{}_\varrho \wedge \Gamma^\varrho{}_\nu =$$

$$= -\frac{K}{2}\left(dy_\nu \wedge \omega^\mu - dy^\mu \wedge \omega_\nu\right) + \frac{K^2}{4}\left(+\underline{y_\nu y_\varrho \omega^\varrho \wedge \omega^\mu} + y^\mu y_\varrho \omega^\varrho \wedge \omega_\nu\right)$$

$$+ \frac{K^2}{4}\left(\underline{y_\varrho}\,\omega^\mu - y^\mu \omega_\varrho\right)\wedge\left(\underline{y_\nu}\,\omega^\varrho - y^\varrho \omega_\nu\right) =$$

$$= \cancel{\tfrac{K}{2}}\left(1+\frac{K}{4}|y|^2\right)\left(\omega^\mu \wedge \omega_\nu - \cancel{\omega_\nu \wedge \omega^\mu}\right) + \frac{K^2}{4}\left(-|y|^2 \omega^\mu \wedge \omega_\nu\right)$$

$$= K\,\omega^\mu \wedge \omega_\nu$$

$$\Omega^\mu{}_\nu = K\, g^\mu{}_\alpha\, g_{\nu\beta}\, \omega^\alpha \wedge \omega^\beta =$$

$$= K\, g^\mu{}_{[\alpha}\, g_{\beta]\nu}\, \omega^\alpha \wedge \omega^\beta = \frac{1}{2} R^\mu{}_{\nu\alpha\beta}\, \omega^\alpha \wedge \omega^\beta$$

$$\Rightarrow \boxed{R^\mu{}_{\nu\alpha\beta} = K\left(g^\mu{}_\alpha\, g_{\beta\nu} - g^\mu{}_\beta\, g_{\alpha\nu}\right)}$$

$$\boxed{R_{\nu\beta} = K(n-1)\, g_{\nu\beta}}$$

$$\boxed{R = K(n-1)\cdot n} \quad \xrightarrow{n=2} \quad \Downarrow \quad R = 2K$$

Gauss curvature!

$\Downarrow$

note that

$$\underline{C^\mu{}_{\nu\rho\sigma} \equiv 0\ !}$$

Also $\underline{\nabla_\rho R^\mu{}_{\nu\alpha\beta} = 0!}$